FLOW AND HEAT EXCHANGE IN A LAYER OF A VISCOUS CONDUCTING FLUID BETWEEN ROTATING PLATES WITH HORIZONTAL GRADIENTS OF TEMPERATURE IN A TRANSVERSE MAGNETIC FIELD

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With the development of technology and intensification of astrophysical and geophysical investigations in which thermal convection plays a significant role, it has become urgent to study convection in the rotating layers of a nonconducting [1, 2] and an electricity-conducting fluid in the presence of a transverse magnetic field [3]. As a rule, no consideration is given to the presence of horizontal temperature gradients that usually arise in real problems and serve frequently as the basic cause of motion [4]. The simplest case of convection in a layer, engendered by a horizontal temperature gradient, without consideration of rotation and of the magnetic field, has been investigated in [5].

We examine the infinite horizontal layer of a viscous conducting fluid of thickness D, bounded by conducting plates at which various constant temperature gradients arbitrarily oriented in the plane of the layer are maintained. The layer is set into rotation at a constant angle of velocity $\mathbf{f}/2$ relative to the z axis, directed across the layer. A constant magnetic field $\mathbf{B}_0$ is applied along the z axis, as is the vector $\mathbf{g}$ of the acceleration of free fall. In the Boussinesq approximation in the equations of motion in a rotating system of coordinates connected to the boundaries of the layer have the form [4]

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p + (\mathbf{B}\nabla)\mathbf{B} + \nu\Delta\mathbf{v} - \mathbf{f}\times\mathbf{v} + \mathbf{g}\alpha T,$$
$$\partial\mathbf{B}/\partial t + (\mathbf{v}\nabla)\mathbf{B} = (\mathbf{B}\nabla)\mathbf{v} + \nu_m\Delta\mathbf{B},\ \partial T/\partial t + (\mathbf{v}\nabla)T = \chi\Delta T,$$
$$\operatorname{div}\mathbf{v} = 0,\ \operatorname{div}\mathbf{B} = 0,$$

where the hydrostatic, magnetic, and centrifugal components have been taken into consideration in the pressure p; $\mathbf{B}$ is the magnetic field, reduced to the dimensions of velocity; α is the coefficient of fluid temperature expansion; $\nu_m = 1/\sigma\mu\mu_0$ is the coefficient of magnetic viscosity. The remaining notation is standard.

For purposes of making these quantities dimensionless, as the units of length, time, velocity, temperature, pressure, and of the magnetic field, we have, respectively, selected D, D^2/ν, and $g\alpha A_1 D^3/\nu$ (A_1 is the temperature gradient at the lower plane), A_1D, $g\alpha A_1 D^2$ (here we take into consideration the fact that the pressure has been normalized with respect to density), B_0. The dimensionless system is written as follows:

$$\begin{gathered}\partial\mathbf{v}/\partial t + G(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \Delta\mathbf{v} + \mathrm{Ha}^2G^{-1}\beta^{-1}(\mathbf{B}\nabla)\mathbf{B} - \mathrm{Ta}^{1/2}(\boldsymbol{\gamma}\times\\ \times\mathbf{v}) + \boldsymbol{\gamma}T,\\ \partial\mathbf{B}/\partial t + G(\mathbf{v}\nabla)\mathbf{B} = G(\mathbf{B}\nabla)\mathbf{v} + \beta^{-1}\Delta\mathbf{B},\\ \partial T/\partial t + G(\mathbf{v}\nabla)T = \mathrm{Pr}^{-1}\Delta T,\ \operatorname{div}\mathbf{v} = 0,\ \operatorname{div}\mathbf{B} = 0\end{gathered} \tag{1}$$

[the Grashof number is $G = g\alpha A_1 D^4/\nu^2$; the Hartmann number is $\mathrm{Ha} = B_0 D/\sqrt{\nu\nu_m}$; the Taylor number is $\mathrm{Ta} = (fD^2/\nu)^2$; the Prandtl number is $\mathrm{Pr} = \nu/\chi$; the Batchelor number is $\beta = \nu/\nu_m$; γ is the unit vector directed along the z axis].

The formulation of the problem leads to the following boundary conditions:

$$v_x = v_y = v_z = 0 \text{ for } z = 0,1,\quad \int_0^1 v_x\,dz = Q,\ \int_0^1 v_y\,dz = 0;$$

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for temperature

$$T = x\cos\varphi_1 + y\sin\varphi_1 \text{ when } z = 0,$$
$$T = A(x\cos\varphi_2 + y\sin\varphi_2) + T_0 \text{ when } z = 1;$$

for the magnetic field [3, 6]

$$\sigma_1\lambda_1(dB_{x,y}/dz) - \sigma B_{x,y} = 0 \text{ when } z = 0,$$
$$\sigma_2\lambda_2(dB_{x,y}/dz) + \sigma B_{x,y} = 0 \text{ when } z = 1,$$

where the angles φ_1 and φ_2 are reckoned from the x axis and in the xy plane give the directions of the temperature gradients at the lower and upper planes, respectively; $A = A_2/A_1$ is the ratio of the upper temperature gradient to the lower temperature gradient; σ_1, λ_1, σ_2, λ_2 are the electrical conductivity and thickness of the lower and upper plates; σ is the electrical conductivity of the fluid; Q is the rate of flow through a section of the layer in the direction of the x axis.

In order to find the solution the velocity and the magnetic field must be represented as functions dependent exclusively on the transverse coordinate z. This allows us significantly to simplify the system of equations (1) and to obtain steady precise solutions of the system for the velocity, the magnetic field, and the temperature, satisfying all boundary conditions.

The exact solution for velocity has the form

$$v_{x,y} = V_{x,y} + 2u_{x,y}z + w_{x,y},\ v_z = 0.$$

Here $V_{x,y}$ are the integration constants, determined from the integral conditions imposed on velocity:

$$V_{x,y} = R_{x,y}/R,\ R = 1 + 4(k_1\gamma_1 - k_2\gamma_2) + 4(k_1^2 + k_2^2)(\gamma_1^2 + \gamma_2^2),$$
$$R_x = (Q - \bar{u}_x)(1 + 2(k_1\gamma_1 - k_2\gamma_2)) + 2\bar{u}_y(k_1\gamma_2 + k_2\gamma_1) -$$
$$- 2u_x(1)[(k_1\gamma_1 - k_2\gamma_2) + 2(k_1^2 + k_2^2)(\gamma_1^2 + \gamma_2^2)] - 2u_y(1)(k_1\gamma_2 + k_2\gamma_1),$$
$$R_y = (Q - \bar{u}_x)2(k_1\gamma_2 + k_2\gamma_1) - \bar{u}_y(1 + 2(k_1\gamma_1 - k_2\gamma_2)) +$$
$$+ 2u_x(1)(k_1\gamma_2 + k_2\gamma_1) - 2u_y(1)[(k_1\gamma_1 - k_2\gamma_2) + 2(k_1^2 + k_2^2)(\gamma_1^2 + \gamma_2^2)],$$
$$\bar{u}_{x,y} = \int_0^1 u_{x,y}\,dz,\quad \gamma_{1,2} = 2(\varepsilon_{1,2} - \varepsilon_{5,4})/(\mathrm{Ta} + \mathrm{Ha}^4)^{1/2},\quad k_{1,2} =$$
$$= [\pm\mathrm{Ha}^2 + (\mathrm{Ta} + \mathrm{Ha}^4)^{1/2}]^{1/2}/\sqrt{2},$$
$$\varepsilon = 2\,\mathrm{ch}\,(2k_1) - 2\cos(2k_2),\ \varepsilon\varepsilon_1 = 2\,\mathrm{sh}\,k_1\cos k_2,\ \varepsilon\varepsilon_2 = 2\,\mathrm{ch}\,k_1\sin k_2,$$
$$\varepsilon\varepsilon_3 = \exp(2k_1) - \cos(2k_2),\ \varepsilon\varepsilon_4 = \sin(2k_2),\ \varepsilon\varepsilon_5 = \mathrm{sh}\,(2k_1),$$
$$u_x = -[\mathrm{Ha}^2((\partial T/\partial x) + \cos\varphi_1) + \mathrm{Ta}^{1/2}((\partial T/\partial y) + \sin\varphi_1)]/[4(\mathrm{Ta} + \mathrm{Ha}^4)],$$
$$u_y = -[\mathrm{Ha}^2((\partial T/\partial y) + \sin\varphi_1) - \mathrm{Ta}^{1/2}((\partial T/\partial x) + \cos\varphi_1)]/[4(\mathrm{Ta} + \mathrm{Ha}^4)],$$
$$\partial T/\partial x = (A\cos\varphi_2 - \cos\varphi_1)z + \cos\varphi_1,$$
$$\partial T/\partial y = (A\sin\varphi_2 - \sin\varphi_1)z + \sin\varphi_1,$$
$$w_{x,y} = \exp[-k_1(1 - z)][C_{1,2}\cos k_2(1 - z) \pm C_{2,1}\sin k_2(1 - z)] +$$
$$+ \exp[-k_1 z][C_{3,4}\cos k_2 z \pm C_{4,3}\sin k_2 z],$$
$$C_{1,2} = V_{x,y}(\varepsilon_1 - \varepsilon_3) \pm V_{y,x}(\varepsilon_2 - \varepsilon_4) - 2u_{x,y}(1)\,\varepsilon_3 \mp 2u_{y,x}(1)\varepsilon_4,$$
$$C_{3,4} = V_{x,y}(\varepsilon_1 - \varepsilon_3) \pm V_{y,x}(\varepsilon_2 - \varepsilon_4) + 2u_{x,y}(1)\varepsilon_1 \pm 2u_{y,x}(1)\varepsilon_2.$$

The solutions for the magnetic field and temperature can easily be found by substituting the expression for velocity into the equations given in [7]. In the present study we have derived expressions for the magnetic field and for temperature, but we do not present them here because of their cumbersome nature. In corresponding limit cases these become the familiar Birikh solutions [5] (Ha = Ta = φ_1 = φ_2 = 0, A = 1), the Eckman solution (G = Ha = 0), the Hartmann solution (G = Ta = 0), and with G = 0 they change to the solutions from [3].

The properties of the exact solution are dealt with on the example of mercury at T ~ 300 K, aluminum at T ~ 900-1000 K, and sodium at T ~ 370 K. All of the graphs have been constructed for Q = φ_1 = φ_2 = 0, A = 1, and the thickness of the layer was assumed to be

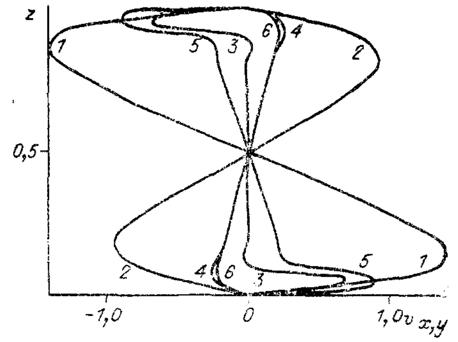

Fig. 1

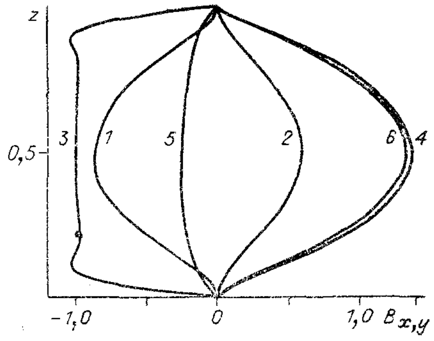

Fig. 2

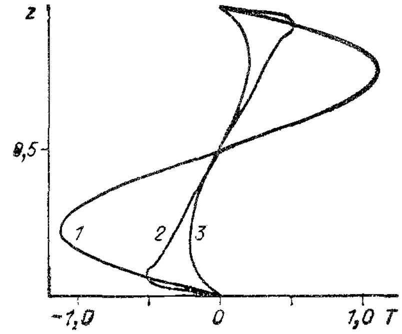

Fig. 3

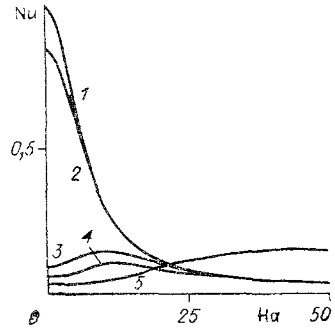

Fig. 4

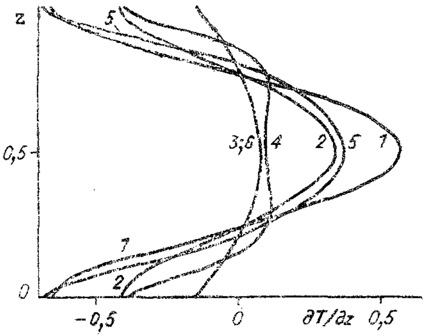

Fig. 5

equal to 1 cm. The Grashof numbers for each substance are calculated for a temperature gradient equal to 1 K/m. In the expression for velocity the two first terms form the core of the flow and the third corresponds to the boundary layer. This is clearly seen in Fig. 1, where the profiles of the x-th (odd numerals) and y-th (even numerals) components of velocity for mercury are shown. Profile 1: $v_x \cdot 10^2$; and profile 2: $v_y \cdot 10^5$, were calculated for Ta = 3.04, Ha = 14.5, which corresponds to a rotational angular velocity of 0.001 rps and a magnetic-field strength of 50 A/m. The core of the flow is linearly dependent on the transverse coordinate z, the thickness of the boundary layers tends toward zero with an increase in the magnetic field, and the y-th components of velocity appear only as a result of rotation and depend significantly on Ta. Profile 3: $v_x \cdot 10^4$, and profile 4: $v_y \cdot 10^3$, were calculated for Ta = $3.04 \cdot 10^6$, Ha = 0.029 (1 rps, 0.1 A/m) (the case of a weak magnetic field and of strong rotation). The fluid is virtually immobile at the core, and we see a periodic shift in the direction of velocity with the flow intensity attenuating toward the center. The Eckman boundary layers can clearly be seen, and their thickness tends toward zero as Ta increases. A linear relationship between the profile of the flow core and the transverse coordinate is characteristic of the y-th component. Profile 5: $v_x \cdot 10^4$, and profile 6: $v_y \cdot 10^3$, have been obtained for Ta = $3.04 \cdot 10^6$, Ha = 14.5 (the case of a strong field and strong rotation). The influence of each of these factors is followed clearly on the profile of the x-th component of velocity, and we see the linear profile of the core and the Ekman boundary layers.

Figure 2 shows the profiles of the x-th and y-th components of the magnetic field: 1) $B_x \cdot 10^7$; 2) $B_y \cdot 10^9$; 3) $B_x \cdot 10^9$; 4) $B_y \cdot 10^7$; 5) $B_x \cdot 10^8$; 6) $B_y \cdot 10^7$. The numbers at the profiles correspond to the same Hartmann and Taylor numbers as the numbers at the velocity profiles. The most interesting case is the one involving strong rotation and a weak magnetic field, and we have reference here to profile 3. The plane rate in the profile of the x-th magnetic-field component indicates that the flow core is virtually nonmoving.

Figure 3 shows the temperature profiles for the same parameters as the profiles of velocity and of the magnetic field.

It would be interesting to look at the change of the flow of heat through the boundary of the layer insofar as this relates to rotation and the transverse magnetic field. In order to study the exchange of heat through the boundary of the layer, let us define the local Nusselt number Nu at the point having coordinates x, y, z as the ratio of the dimensionless temperature gradient across the layer to the transverse gradient in the absence of rotation and in the absence of a magnetic field: $Nu = (\partial T/\partial z)/(\partial T/\partial z)|_{Ha,Ta=0}$ (Nu depends on Q, G, Pr, Ha, Ta).

Figure 4 shows Nu at the boundary in the case of x = y = z = 0 as a function of G for various Ta, for a variety of substances. Curve 1 corresponds to the Nu functions for aluminum and sodium when Ta = 2.7 and Ta = 5.2 (0.01 rps); curve 2 shows this relationship for mercury in the case of Ta = 304 (0.01 rps); curve 3 shows this relationship for aluminum in the case of $Ta = 2.7 \cdot 10^4$ (1 rps); curve 4 is the same for sodium at $Ta = 5.2 \cdot 10^4$ (1 rps); curve 5 represents $Nu \cdot 10$ for mercury when $Ta = 3.04 \cdot 10^6$ (1 rps). We can see that the intensity of the heat flux in the case of weak rotation diminishes sharply with an increase in the magnetic field. In the case of strong rotation we observe a displacement of the heat-flux maximum in the direction of strong magnetic fields. With an increase in the magnetic field the flow of heat tends toward a limit that is independent of Ta.

Figure 5 shows the relationship for the transverse temperature gradient through a section of the layer for liquid sodium. In the case of equal temperature gradients at the boundaries, the integral heat flow across the layer is equal to zero for any G and Ta. All of the curves are symmetrical relative to the middle of the layer. The numerals at the curves correspond to the following parameters: 1) $(\partial T/\partial z) \cdot 10^3$, Ta = Ha = 0; 2) $(\partial T/\partial z) \cdot 10^3$, Ta = 5.2, Ha = 5; 3) $(\partial T/\partial z) \cdot 10^4$, Ta = 5.2, Ha = 50; 4) $(\partial T/\partial z) \cdot 10^4$, $Ta = 5.2 \cdot 10^4$, Ha = 1; 5) $(\partial T/\partial z) \cdot 10^4$, $Ta = 5.2 \cdot 10^4$, Ha = 10; 6) $(\partial T/\partial z) \cdot 10^4$, $Ta = 5.2 \cdot 10^4$, Ha = 50.

The graphs show that large changes in Nu are characteristic of small changes in G, especially in the case of small Ta. It thus becomes possible to control the flow of heat at the boundary of the layer. This feature of this problem is governed by the rapid restructuring of the boundary layers of the flow with slight changes in G and Ta.

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